

On some Algebraic Properties for q -Meixner Multiple Orthogonal Polynomials of the First Kind

Jorge Arvesú* and A.M. Ramírez-Aberasturis
 Department of Mathematics, Universidad Carlos III de Madrid,
 Avenida de la Universidad, 30, 28911, Leganés, Spain

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Abstract

We study a new family of q -Meixner multiple orthogonal polynomials of the first kind. The discrete orthogonality conditions are considered over a non-uniform lattice with respect to different q -analogues of Pascal distributions. We address some algebraic properties, namely raising and lowering operators as well as Rodrigues-type. Based on the explicit expressions for the raising and lowering operator a high-order linear q -difference equation with polynomial coefficients for the q -Meixner multiple orthogonal polynomials of the first kind is obtained. Finally, we obtain the nearest neighbor recurrence relation based on a purely algebraic approach.

Keywords: multiple orthogonal polynomials; Hermite–Padé approximation; difference equations; classical orthogonal polynomials of a discrete variable; Meixner polynomials; q -polynomials

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1 Introduction

We begin by introducing the basic background materials. Let $\vec{\mu} = (\mu_1, \dots, \mu_r)$ be a vector of r positive discrete measures (with finite moments)

$$\mu_i = \sum_{k=0}^{N_i} \omega_{i,k} \delta_{x_{i,k}}, \quad \omega_{i,k} > 0, \quad x_{i,k} \in \mathbb{R}, \quad N_i \in \mathbb{N} \cup \{+\infty\}, \quad i = 1, 2, \dots, r,$$

where $\delta_{x_{i,k}}$ denotes the Dirac delta function and $x_{i_1,k} \neq x_{i_2,k}$, $k = 0, \dots, N_i$, whenever $i_1 \neq i_2$. By $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ we denote a multi-index, where \mathbb{N} stands for the set of all nonnegative integers. A type II discrete multiple orthogonal polynomial $P_{\vec{n}}(x)$, corresponding to the multi-index \vec{n} , is a polynomial of degree $\leq |\vec{n}| = n_1 + \dots + n_r$ which satisfies the orthogonality conditions [3]

$$\sum_{k=0}^{N_i} P_{\vec{n}}(x_{i,k}) x_{i,k}^j \omega_{i,k} = 0, \quad j = 0, \dots, n_i - 1, \quad i = 1, \dots, r. \quad (1.1)$$

The orthogonality conditions (1.1) give a linear system of $|\vec{n}|$ homogeneous equations for the $|\vec{n}| + 1$ unknown coefficients of $P_{\vec{n}}(x)$. This polynomial solution $P_{\vec{n}}$ always exists. We restrict our attention to a unique solution (up to a multiplicative factor) with $\deg P_{\vec{n}}(x) = |\vec{n}|$. If this happen for every multi-index \vec{n} , we say that \vec{n} is normal [11]. If the above system of measures forms an AT system [3, 11] then every multi-index is normal. In this paper we will deal with such system of discrete measures.

In [3] some type II discrete multiple orthogonal polynomials on the linear lattice $x(s) = s$ were considered. Moreover, multiple Meixner polynomials (of first and second kind, respectively) were studied. Indeed, the monic multiple Meixner polynomials of the first kind [3] $M_{\vec{n}}^{\vec{\alpha}, \beta}(x)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$, satisfy the following orthogonality conditions with different positive parameters $\alpha_1, \dots, \alpha_r \in (0, 1)$ (indexed by $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$) and $\beta > 0$

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{\vec{\alpha}, \beta}(x) (-x)_j v^{\alpha_i, \beta}(x) = 0, \quad j = 0, \dots, n_i - 1, \quad i = 1, \dots, r,$$

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where

$$v^{\alpha_i, \beta}(x) = \begin{cases} \frac{\Gamma(\beta+x)}{\Gamma(\beta)} \frac{\alpha_i^x}{\Gamma(x+1)}, & \text{if } x \in \mathbb{R} \setminus (\mathbb{Z}_- \cup \{-\beta, -\beta-1, \beta-2, \dots\}), \\ 0, & \text{otherwise,} \end{cases}$$

and $(x)_j = (x)(x+1) \cdots (x+j-1)$, $(x)_0 = 1$, $j \geq 1$, denotes the Pochhammer symbol. Notice that the multi-index $\vec{n} \in \mathbb{N}^r$ is normal whenever $0 < \alpha_i < 1$, $i = 1, 2, \dots, r$, and with all the α_i different (see [3]).

Furthermore, it was found the following raising operators

$$\mathcal{R}_{\vec{n}}^{\alpha_i, \beta} [M_{\vec{n}}^{\vec{\alpha}, \beta}(x)] = -M_{\vec{n}+\vec{e}_i}^{\vec{\alpha}, \beta-1}(x), \quad i = 1, \dots, r, \quad (1.2)$$

where

$$\mathcal{R}_{\vec{n}}^{\alpha_i, \beta} = \frac{\alpha_i(\beta-1)}{(1-\alpha_i)v^{\alpha_i, \beta-1}(x)} \nabla v^{\alpha_i, \beta}(x),$$

and $\nabla f(x) = f(x) - f(x-1)$ denotes the backward difference operator. As a consequence of (1.2) the following Rodrigues-type formula

$$M_{\vec{n}}^{\vec{\alpha}, \beta}(x) = (\beta)_{|\vec{n}|} \prod_{j=1}^r \left(\frac{\alpha_j}{\alpha_j-1} \right)^{n_j} \frac{\Gamma(\beta)\Gamma(x+1)}{\Gamma(\beta+x)} \mathcal{M}_{\vec{n}}^{\vec{\alpha}} \left(\frac{\Gamma(\beta+|\vec{n}|+x)}{\Gamma(\beta+|\vec{n}|)\Gamma(x+1)} \right), \quad (1.3)$$

where

$$\mathcal{M}_{\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^r (\alpha_i^{-x} \nabla^{n_i} \alpha_i^x),$$

was obtained.

Moreover, two important algebraic properties were deduced for multiple Meixner polynomials of the first kind [3], namely the $(r+1)$ -order linear difference equation [9]

$$\prod_{i=1}^r \mathcal{R}_{\vec{n}}^{\alpha_i, \beta} \triangle M_{\vec{n}}^{\vec{\alpha}, \beta}(x) + \sum_{i=1}^r n_i \prod_{\substack{j=1 \\ j \neq i}}^r \mathcal{R}_{\vec{n}}^{\alpha_j, \beta} M_{\vec{n}}^{\vec{\alpha}, \beta}(x) = 0, \quad (1.4)$$

where $\triangle f(x) = f(x+1) - f(x)$, and the recurrence relation [3, 8]

$$\begin{aligned} x M_{\vec{n}}^{\vec{\alpha}, \beta}(x) &= M_{\vec{n}+\vec{e}_k}^{\vec{\alpha}, \beta}(x) + \left((\beta+|\vec{n}|) \left(\frac{\alpha_k}{1-\alpha_k} \right) + \sum_{i=1}^r \frac{n_i}{1-\alpha_i} \right) M_{\vec{n}}^{\vec{\alpha}, \beta}(x) \\ &+ \sum_{i=1}^r \frac{\alpha_i n_i (\beta+|\vec{n}|-1)}{(\alpha_i-1)^2} M_{\vec{n}-\vec{e}_i}^{\vec{\alpha}, \beta}(x), \end{aligned} \quad (1.5)$$

where the multi-index \vec{e}_i is the standard r dimensional unit vector with the i -th entry equals 1 and 0 otherwise.

The multiple Meixner polynomials of the first kind $M_{\vec{n}, \beta}^{\vec{\alpha}}(x)$ are common eigenfunctions of the above two linear difference operators of order $(r+1)$, given by (1.4) and (1.5), respectively.

In this paper we will introduce a q -analogue of such multiple orthogonal polynomials, i.e. when the component measures of $\vec{\mu}$ are different q -Poisson distributions and study the aforementioned algebraic properties. Our goal is to continue the recent investigations in [2, 5, 12] for some families of q -multiple orthogonal polynomials regarding their algebraic properties. In [4, 9, 13] an $(r+1)$ -order difference equation for some discrete multiple orthogonal polynomials was obtained. Furthermore, the explicit expressions for the coefficients of $(r+2)$ -term recurrence relations are a very important issue for the study of some type of asymptotic behaviors for discrete multiple orthogonal polynomials. In [1] the weak asymptotics was studied for multiple Meixner polynomials of the first and second kind, respectively. The zero distribution of multiple Meixner polynomials was also studied. Another interesting fact involving the knowledge of the $(r+2)$ -term recurrence relations is the attainment of a Christoffel–Darboux kernel [6] among other applications, which plays important role in correlation kernel as in the unitary random matrix model with external source.

The content of this paper is as follows. In Section 2 we define the q -Meixner multiple orthogonal polynomials of the first kind. Moreover, we will prove that these multiple orthogonal polynomials can be explicitly expressed by means of Rodrigues-type formula. This fact provides the background materials for the next Section 3, in which we obtain the $(r+1)$ -order q -difference equation, with polynomial coefficients on a non-uniform lattice $x(s)$. Finally, in Section 4 the nearest neighbor recurrence relation is obtained (an $(r+2)$ -term recurrence relations). Explicit expressions for the recurrence coefficients are given. The paper ends by summarising our findings in Section 5.

2 q -Meixner multiple orthogonal polynomials of the first kind

Aimed to define a new family of q -multiple orthogonal polynomials [2] let us consider the following r positive discrete measures on \mathbb{R}^+ ,

$$\mu_i = \sum_{s=0}^{\infty} \omega_i(k) \delta(k-s), \quad \omega_i > 0, \quad i = 1, 2, \dots, r. \quad (2.1)$$

Here $\omega_i(s) = v_q^{\alpha_i, \beta}(s) \triangle x(s-1/2)$, $x(s) = (q^s - 1)/(q - 1)$, and

$$v_q^{\alpha_i, \beta}(s) = \begin{cases} \frac{\alpha_i^s}{\Gamma_q(s+1)} \frac{\Gamma_q(\beta+s)}{\Gamma_q(\beta)}, & \text{if } s \in \mathbb{R}^+ \cup \{0\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < \alpha_i < 1$, $\beta > 0$, $i = 1, 2, \dots, r$, and with all the α_i different. Recall that the q -Gamma function is given by

$$\Gamma_q(s) = \begin{cases} f(s; q) = (1-q)^{1-s} \frac{\prod_{k \geq 0} (1-q^{k+1})}{\prod_{k \geq 0} (1-q^{s+k})}, & 0 < q < 1, \\ q^{\frac{(s-1)(s-2)}{2}} f(s; q^{-1}), & q > 1. \end{cases}$$

See also [7, 10] for the above definition of the q -Gamma function.

Lemma 2.1. [5] *The system of functions*

$$\alpha_1^s, x(s)\alpha_1^s, \dots, x(s)^{n_1-1}\alpha_1^s, \dots, \alpha_r^s, x(s)\alpha_r^s, \dots, x(s)^{n_r-1}\alpha_r^s,$$

with $\alpha_i > 0$, $i = 1, 2, \dots, r$, and $(\alpha_i/\alpha_j) \neq q^k$, $k \in \mathbb{Z}$, $i, j = 1, \dots, r$, $i \neq j$, forms a Chebyshev system on \mathbb{R}^+ for every $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$.

As a consequence of Lemma 2.1 the system of measures $\mu_1, \mu_2, \dots, \mu_r$ given in (2.1) forms an AT system on \mathbb{R}^+ . For this system of measures we will define a new family of q -multiple orthogonal polynomials.

Recall that the q -analogue of the Stirling polynomials denoted by $[s]_q^{(k)}$, is a polynomial of degree k in the variable $x(s) = (q^s - 1)/(q - 1)$, i.e.

$$[s]_q^{(k)} = \prod_{j=0}^{k-1} \frac{q^{s-j} - 1}{q - 1} = x(s)x(s-1) \cdots x(s-k+1) \quad \text{for } k > 0, \quad \text{and} \quad [s]_q^{(0)} = 1.$$

Moreover, $[s]_q^{(k)} = q^{-\binom{k}{2}} x^k(s) + \text{lower terms} = \mathcal{O}(q^{ks})$, where $\mathcal{O}(\cdot)$ stands for the big-O notation. Observe that, when q goes to 1, the symbol $[s]_q^{(k)}$ converges to $(-1)^k (-s)_k$, where $(s)_k$ is the Pochhammer symbol.

Definition 2.2. A polynomial $M_{q, \vec{n}}^{\vec{\alpha}, \beta}(s)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ that verifies the orthogonality conditions

$$\sum_{s=0}^{\infty} M_{q, \vec{n}}^{\vec{\alpha}, \beta}(s) [s]_q^{(k)} v_q^{\alpha_i, \beta}(s) \triangle x(s-1/2) = 0, \quad 0 \leq k \leq n_i - 1, \quad i = 1, \dots, r, \quad (2.2)$$

(with respect to the measures (2.1)) is said to be the q -Meixner multiple orthogonal polynomial of the first kind.

In the sequel, we will use the following difference operators

$$\Delta \stackrel{\text{def}}{=} \frac{\triangle}{\triangle x(s-1/2)}, \quad (2.3)$$

$$\nabla^{n_i} = \underbrace{\nabla \cdots \nabla}_{n_i \text{ times}}, \quad \text{where} \quad \nabla \stackrel{\text{def}}{=} \frac{\nabla}{\nabla x(s+1/2)},$$

and $\nabla x_1(s) \stackrel{\text{def}}{=} \nabla x(s+1/2) = \triangle x(s-1/2) = q^{s-1/2}$. Moreover, we will only consider monic q -Meixner multiple orthogonal polynomials of the first kind.

Proposition 2.3. *There holds the following q -analogue of Rodrigues-type formula*

$$M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \mathcal{K}_q^{\vec{n},\vec{\alpha},\beta} \frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)} \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} \left(\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \right), \quad (2.4)$$

where

$$\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^r \mathcal{M}_{q,n_i}^{\alpha_i}, \quad \mathcal{M}_{q,n_i}^{\alpha_i} = (\alpha_i)^{-s} \nabla^{n_i} (\alpha_i q^{n_i})^s, \quad (2.5)$$

and

$$\mathcal{K}_q^{\vec{n},\vec{\alpha},\beta} = (-1)^{|\vec{n}|} [-\beta]_q^{(|\vec{n}|)} q^{-\frac{|\vec{n}|}{2}} \left(\prod_{i=1}^r \frac{\alpha_i^{n_i} \prod_{j=1}^{n_i} q^{|\vec{n}|_i + \beta + j - 1}}{\prod_{j=1}^{n_i} (\alpha_i q^{|\vec{n}|_i + \beta + j - 1} - 1)} \right) \left(\prod_{i=1}^r q^{n_i \sum_{j=i}^r n_j} \right),$$

where $|\vec{n}|_i = n_1 + \dots + n_{i-1}$, $|\vec{n}|_1 = 0$.

Proof. We first start by finding the raising operators. Thus, we substitute $[s]_q^{(k)}$ in (2.2) by the following finite-difference expression

$$[s]_q^{(k)} = \frac{q^{(k-1)/2}}{[k+1]_q^{(1)}} \nabla [s+1]_q^{(k+1)},$$

i.e.,

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \nabla [s+1]_q^{(k+1)} v_q^{\alpha_i}(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_i - 1, \quad i = 1, \dots, r.$$

Then, using summation by parts along with conditions $v_q^{\alpha_i,\beta}(-1) = v_q^{\alpha_i,\beta}(\infty) = 0$, we get r raising operators

$$\mathcal{D}_q^{\alpha_i,\beta} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = -q^{1/2} M_{q,\vec{n}+\vec{e}_i}^{\vec{\alpha}_{i,1/q},\beta-1}(s), \quad i = 1, \dots, r, \quad (2.6)$$

where

$$\mathcal{D}_q^{\alpha_i,\beta} = \left(\frac{\alpha_i x (\beta - 1) q^{|\vec{n}|}}{(1 - \alpha_i q^{|\vec{n}|+\beta-1}) v_q^{\alpha_i/q,\beta-1}(s)} \nabla v_q^{\alpha_i,\beta}(s) \right),$$

$$\vec{\alpha}_{i,1/q} = (\alpha_1, \dots, \alpha_i/q, \dots, \alpha_r).$$

Indeed,

$$\begin{aligned} q^{-|\vec{n}|-1/2} \mathcal{D}_q^{\alpha_i,\beta} f(s) &= \frac{1}{1 - \alpha_i q^{|\vec{n}|+\beta-1}} \left[\alpha_i q^{\beta-1} (x(s) - x(1-\beta)) - x(s) \right] f(s) \\ &+ \frac{1}{1 - \alpha_i q^{|\vec{n}|+\beta-1}} x(s) \nabla f(s), \end{aligned}$$

for any function $f(s)$ defined on the discrete variable s . Observe that $\mathcal{D}_q^{\alpha_i,\beta}$ raises by 1 the i -th component of the multi-index \vec{n} in (2.6).

Finally, using the raising operators (2.6) in a recursive way one obtains the Rodrigues-type formula (2.4). \square

Remark 2.4. *Since we are dealing with an AT-system of positive discrete measures (2.1), then the q -Meixner multiple orthogonal polynomial of the first kind $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ has exactly $|\vec{n}|$ different zeros on \mathbb{R}^+ (see [3, Theorem 2.1, pp. 26–27]).*

3 High-order q -difference equation

The strategy that we will follow to deduce the $(r+1)$ -order q -difference equation is the following.

First step. Define an r -dimensional subspace \mathbb{V} of polynomials on the variable $x(s)$ of degree at most $|\vec{n}| - 1$ by means of interpolatory conditions.

Second step. Find the lowering operator and express its action on $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ as a linear combination of the basis elements of \mathbb{V} .

Third step. Combine the lowering and the raising operators (2.6) to get an $(r+1)$ -order q -difference equation (in the same fashion that [4], [5], and [9]).

These steps represent the general features of the algebraic approach we are using in this section, however some ad hoc' computations are needed because of the dependence of the explicit expressions involved in the above steps on the given family of multiple orthogonal polynomials.

Lemma 3.1. *Let \mathbb{V} be the linear subspace of polynomials $Q(s)$ on the lattice $x(s)$ of degree at most $|\vec{n}| - 1$ defined by the following conditions*

$$\sum_{s=0}^{\infty} Q(s) [s]_q^{(k)} v_q^{q\alpha_j, \beta+1}(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2 \quad \text{and} \quad j = 1, \dots, r.$$

Then, the system $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}, \beta+1}(s)\}_{i=1}^r$, where $\vec{\alpha}_{i,q} = (\alpha_1, \dots, q\alpha_i, \dots, \alpha_r)$, is a basis for \mathbb{V} .

Proof. Considering the orthogonality relations

$$\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}, \beta+1}(s) [s]_q^{(k)} v_q^{q\alpha_j, \beta+1}(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2, \quad j = 1, \dots, r,$$

we have that polynomials $M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}, \beta+1}(s)$, $i = 1, \dots, r$, belong to \mathbb{V} .

Let proceed by 'reductio ad absurdum' and assume that there exists constants λ_i , $i = 1, \dots, r$, such that

$$\sum_{i=1}^r \lambda_i M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}, \beta+1}(s) = 0, \quad \text{where} \quad \sum_{i=1}^r |\lambda_i| > 0.$$

Then, multiplying the previous equation by $[s]_q^{(n_k-1)} v_q^{\alpha_k, \beta}(s) \nabla x_1(s)$ and then taking summation on s from 0 to ∞ , one gets

$$\sum_{i=1}^r \lambda_i \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}, \beta+1}(s) [s]_q^{(n_k-1)} v_q^{\alpha_k, \beta}(s) \nabla x_1(s) = 0.$$

Hence, from relations

$$\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}, \beta+1}(s) [s]_q^{(n_k-1)} v_q^{\alpha_k, \beta}(s) \nabla x_1(s) = c\delta_{i,k}, \quad c \in \mathbb{R} \setminus \{0\}, \quad (3.1)$$

we deduce that $\lambda_k = 0$ for $k = 1, \dots, r$. Here $\delta_{i,k}$ represents the Kronecker delta symbol. Therefore, $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}, \beta+1}(s)\}_{i=1}^r$ is linearly independent in \mathbb{V} . Furthermore, we know that any polynomial of \mathbb{V} can be determined with $|\vec{n}|$ coefficients while $(|\vec{n}| - r)$ linear conditions are imposed on \mathbb{V} , consequently the dimension of \mathbb{V} is at most r . Hence, the system $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}, \beta+1}(s)\}_{i=1}^r$ spans \mathbb{V} , which completes the proof. \square

Now we will prove that operator (2.3) is indeed a lowering operator for the sequence of q -Meixner multiple orthogonal polynomials of the first kind $M_{q,\vec{n}}^{\vec{\alpha}, \beta+1}(s)$.

Lemma 3.2. *There holds the following relation*

$$\Delta M_{q,\vec{n}}^{\vec{\alpha}, \beta}(s) = \sum_{i=1}^r q^{|\vec{n}|-n_i+1/2} \frac{1 - \alpha_i q^{n_i+\beta}}{1 - \alpha_i q^{|\vec{n}|+\beta}} [n_i]_q^{(1)} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}, \beta+1}(s). \quad (3.2)$$

Proof. Using summation by parts we have

$$\begin{aligned} \sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha}, \beta}(s) [s]_q^{(k)} v_q^{q\alpha_j, \beta+1}(s) \nabla x_1(s) &= - \sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha}, \beta}(s) \nabla [[s]_q^{(k)} v_q^{q\alpha_j, \beta+1}(s)] \nabla x_1(s) \\ &= - \sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha}, \beta}(s) \varphi_{j,k}(s) v_q^{\alpha_j, \beta}(s) \nabla x_1(s), \end{aligned} \quad (3.3)$$

where

$$\varphi_{j,k}(s) = q^{1/2} \left(\frac{q^\beta x(s)}{x(\beta)} + 1 \right) [s]_q^{(k)} - q^{-1/2} \frac{x(s)}{\alpha_j x(\beta)} [s-1]_q^{(k)},$$

is a polynomial of degree $\leq k+1$ in the variable $x(s)$. Consequently, from the orthogonality conditions (2.2) we get

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) [s]_q^{(k)} v_q^{q\alpha_j,\beta+1}(s) \nabla x_1(s) = 0, \quad 0 \leq k \leq n_j - 2, \quad j = 1, \dots, r.$$

Hence, from Lemma 3.1, $\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \in \mathbb{V}$. Moreover, $\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ can univocally be expressed as a linear combination of polynomials $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\}_{i=1}^r$, i.e.

$$\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \sum_{i=1}^r \beta_i M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s), \quad \sum_{i=1}^r |\beta_i| > 0. \quad (3.4)$$

Multiplying both sides of the equation (3.4) by $[s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \nabla x_1(s)$ and using relations (3.1) one has

$$\begin{aligned} \sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) [s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \nabla x_1(s) &= \sum_{i=1}^r \beta_i \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s) [s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \nabla x_1(s) \\ &= \beta_k \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q},\beta+1}(s) [s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \nabla x_1(s). \end{aligned} \quad (3.5)$$

If we replace $[s]_q^{(k)}$ by $[s]_q^{(n_k-1)}$ in the left-hand side of equation (3.3), then left-hand side of equation (3.5) transforms into relation

$$\begin{aligned} \sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) [s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \nabla x_1(s) &= - \sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \varphi_{k,n_k-1}(s) v_q^{\alpha_k,\beta}(s) \nabla x_1(s) \\ &= \frac{q^{-1/2} (1 - \alpha_k q^{n_k+\beta})}{\alpha_k x(\beta)} \sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) [s]_q^{(n_k)} v_q^{\alpha_k,\beta}(s) \nabla x_1(s). \end{aligned} \quad (3.6)$$

Here we have used that $x(s)[s-1]_q^{(n_k-1)} = [s]_q^{(n_k)}$ to get $\varphi_{k,n_k-1}(s) = -\frac{q^{-1/2}(1-\alpha_k q^{n_k+\beta})}{\alpha_k x(\beta)} [s]_q^{(n_k)} +$ lower terms.

On the other hand, from (2.6) one has that

$$\frac{q^{-1/2} (1 - \alpha_k q^{|\vec{n}|+\beta})}{\alpha_k x(\beta)} v_q^{\alpha_k,\beta}(s) M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = -q^{|\vec{n}|-1/2} \nabla [v_q^{q\alpha_k,\beta+1}(s) M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q},\beta+1}(s)]. \quad (3.7)$$

Then, by conveniently substituting (3.7) in the right-hand side of equation (3.6) and using once more summation by parts, we get

$$\begin{aligned} \sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) [s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \nabla x_1(s) &= -q^{|\vec{n}|-1} \frac{1 - \alpha_k q^{n_k+\beta}}{1 - \alpha_k q^{|\vec{n}|+\beta}} \sum_{s=0}^{\infty} [s]_q^{(n_k)} \nabla [v_q^{q\alpha_k,\beta+1}(s) M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q},\beta+1}(s)] \nabla x_1(s) \\ &= q^{|\vec{n}|-1} \frac{1 - \alpha_k q^{n_k+\beta}}{1 - \alpha_k q^{|\vec{n}|+\beta}} \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q},\beta+1}(s) \Delta [[s]_q^{(n_k)}] v_q^{q\alpha_k,\beta+1}(s) \nabla x_1(s). \end{aligned}$$

Since $\Delta [s]_q^{(n_k)} = q^{3/2-n_k} [n_k]_q^{(1)} [s]_q^{(n_k-1)}$ we finally have

$$\begin{aligned} \sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) [s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \nabla x_1(s) &= q^{|\vec{n}|-n_k+1/2} \frac{1 - \alpha_k q^{n_k+\beta}}{1 - \alpha_k q^{|\vec{n}|+\beta}} [n_k]_q^{(1)} \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{i,q},\beta+1}(s) [s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \nabla x_1(s). \end{aligned}$$

Therefore, comparing this equation with (3.5) we obtain the coefficients in the expansion (3.4), i.e.

$$\beta_k = q^{|\vec{n}|-n_k+1/2} \frac{1 - \alpha_k q^{n_k+\beta}}{1 - \alpha_k q^{|\vec{n}|+\beta}} [n_k]_q^{(1)},$$

which proves relation (3.2). \square

Theorem 3.3. *The q -Meixner multiple orthogonal polynomial of the first kind $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ satisfies the following $(r+1)$ -order q -difference equation*

$$\prod_{i=1}^r \mathcal{D}_q^{\alpha_i,\beta} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = - \sum_{i=1}^r q^{|\vec{n}|-n_i+1} \frac{1 - \alpha_i q^{n_i+\beta}}{1 - \alpha_i q^{|\vec{n}|+\beta}} [n_i]_q^{(1)} \prod_{\substack{j=1 \\ j \neq i}}^r \mathcal{D}_q^{\alpha_j,\beta} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s). \quad (3.8)$$

Proof. Since operators (2.6) are commuting, we write

$$\prod_{i=1}^r \mathcal{D}_q^{\alpha_i,\beta} = \left(\prod_{\substack{j=1 \\ j \neq i}}^r \mathcal{D}_q^{\alpha_j,\beta} \right) \mathcal{D}_q^{\alpha_i,\beta}, \quad (3.9)$$

and then using (2.6), by acting on equation (3.2) with the product of operators (3.9), we obtain the following relation

$$\begin{aligned} \prod_{i=1}^r \mathcal{D}_q^{\alpha_i,\beta} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) &= \sum_{i=1}^r q^{|\vec{n}|-n_i+1/2} \frac{1 - \alpha_i q^{n_i+\beta}}{1 - \alpha_i q^{|\vec{n}|+\beta}} [n_i]_q^{(1)} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \mathcal{D}_q^{\alpha_j,\beta} \right) \mathcal{D}_q^{\alpha_i,\beta} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha},\beta+1}(s) \\ &= - \sum_{i=1}^r q^{|\vec{n}|-n_i+1} \frac{1 - \alpha_i q^{n_i+\beta}}{1 - \alpha_i q^{|\vec{n}|+\beta}} [n_i]_q^{(1)} \prod_{\substack{j=1 \\ j \neq i}}^r \mathcal{D}_q^{\alpha_j,\beta} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s), \end{aligned}$$

which proves (3.8). \square

4 Nearest neighbor recurrence relation

Let us start recalling that for any function $f(s)$ defined on the discrete variable s and a positive integer n_i there holds (see Lemma 5.1 from [5])

$$\mathcal{M}_{q,n_i}^{\alpha_i}(x(s)f(s)) = q^{-n_i+1/2} x(n_i) (\alpha_i)^{-s} \nabla^{n_i-1} (\alpha_i q^{n_i})^s f(s) + \frac{x(s) - x(n_i)}{q^{n_i}} \mathcal{D}_{q,n_i}^{\alpha_i} f(s), \quad (4.1)$$

where difference operator $\mathcal{M}_{q,n_i}^{\alpha_i}$ is given in (2.5).

Now, let us proceed with the nearest neighbor recurrence relation.

Theorem 4.1. *The q -Meixner multiple orthogonal polynomials of the first kind satisfy the following $(r+2)$ -term recurrence relation*

$$\begin{aligned} x(s) \tilde{M}_{q,\vec{n}}^{\vec{\alpha},\beta}(s) &= c_{\vec{n},k} \tilde{M}_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha},\beta}(s) + b_{\vec{n},k} \tilde{M}_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \\ &+ \sum_{i=1}^r x(n_i) \prod_{\substack{j=1 \\ j \neq i}}^r \frac{\alpha_i q^{|\vec{n}|} - \alpha_j q^{n_j}}{\alpha_i q^{n_i} - \alpha_j q^{n_j}} \prod_{i=1}^r \frac{\alpha_i q^{|\vec{n}|+\beta-1} - 1}{\alpha_i q^{|\vec{n}|+\beta+n_i-1} - 1} \frac{\alpha_i q^{n_i} - 1}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \\ &\times \frac{\alpha_i q^{|\vec{n}|+n_i-1} x(\beta + |\vec{n}| - 1)}{\alpha_i q^{|\vec{n}|+\beta+n_i-1} - 1} \frac{1}{\alpha_i q^{|\vec{n}|+\beta+n_i-2} - 1} \tilde{M}_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha},\beta}(s), \end{aligned} \quad (4.2)$$

where $\tilde{M}_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = (\mathcal{K}_q^{\vec{n},\vec{\alpha},\beta})^{-1} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ and

$$\begin{aligned} b_{\vec{n},k} &= \prod_{i=1}^r \frac{\alpha_i q^{|\vec{n}|+\beta} - 1}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \left(\sum_{i=1}^r q^{|\vec{n}|_i} x(n_i) \frac{\alpha_i q^{\sum_{j=1}^r n_j} - 1}{\alpha_i q^{|\vec{n}|+\beta} - 1} + (q-1) q^{|\vec{n}|+\beta} \prod_{i=1}^r \frac{\alpha_i q^{n_i} - 1}{\alpha_i q^{|\vec{n}|+\beta} - 1} \right. \\ &\quad \left. - (q-1) \prod_{i=1}^r x(n_i) \sum_{i=1}^r \prod_{j=1}^i \frac{1}{\alpha_j q^{|\vec{n}|+\beta} - 1} + q^{|\vec{n}|} \frac{(\alpha_k q^{n_k+1}) x(\beta + |\vec{n}|)}{1 - \alpha_k q^{|\vec{n}|+\beta+n_k+1}} \right) \end{aligned}$$

$$\begin{aligned}
& - \sum_{i=1}^r \prod_{j \neq i}^r \frac{\alpha_i q^{|\vec{n}|} - \alpha_j q^{n_j}}{\alpha_i q^{n_i} - \alpha_j q^{n_j}} \frac{x(n_i)(\alpha_i q^{n_i} - 1)}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \frac{\alpha_i q^{\beta+|\vec{n}|+n_i-1}}{\alpha_i q^{|\vec{n}|+\beta+n_i-1} - 1}, \\
c_{\vec{n},k} &= q^{|\vec{n}|-1/2} \prod_{i=1}^r \frac{\alpha_i q^{|\vec{n}|+\beta} - 1}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \frac{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}{\alpha_k q^{|\vec{n}|+\beta+n_k+1} - 1}.
\end{aligned}$$

Proof. Consider equation

$$\begin{aligned}
& (\alpha_k)^{-s} \nabla^{n_k+1} (\alpha_k q^{n_k+1})^s \frac{\Gamma_q(\beta+|\vec{n}|+1+s)}{\Gamma_q(\beta+|\vec{n}|+1)\Gamma_q(s+1)} \\
&= (\alpha_k)^{-s} \nabla^{n_k} \left[q^{-s+1/2} \nabla \left((\alpha_k q^{n_k+1})^s \frac{\Gamma_q(\beta+|\vec{n}|+1+s)}{\Gamma_q(\beta+|\vec{n}|+1)\Gamma_q(s+1)} \right) \right] \\
&= q^{1/2} (\alpha_k)^{-s} \nabla^{n_k} (\alpha_k q^{n_k})^s \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \\
&+ q^{1/2} \frac{\alpha_k q^{|\vec{n}|+\beta+n_k+1} - 1}{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)} (\alpha_k)^{-s} \nabla^{n_k} (\alpha_k q^{n_k})^s x(s) \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)},
\end{aligned}$$

which can be rewritten in terms of difference operators (2.5) as follows

$$\begin{aligned}
& q^{-1/2} \mathcal{M}_{q,n_k+1}^{\alpha_k} \frac{\Gamma_q(\beta+|\vec{n}|+1+s)}{\Gamma_q(\beta+|\vec{n}|+1)\Gamma_q(s+1)} \\
&= \mathcal{M}_{q,n_k}^{\alpha_k} \left(\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \left(1 + \frac{\alpha_k q^{|\vec{n}|+\beta+n_k+1} - 1}{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)} x(s) \right) \right). \quad (4.3)
\end{aligned}$$

Since operators (2.5) are commuting the multiplication of equation (4.3) from the left-hand side by the product $\left(\prod_{\substack{j=1 \\ j \neq k}}^r \mathcal{M}_{q,n_j}^{\alpha_j} \right)$ yields

$$\begin{aligned}
& \frac{\alpha_k q^{|\vec{n}|+\beta+n_k+1} - 1}{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)} \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} x(s) \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \\
&= q^{-1/2} \mathcal{M}_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha}} \frac{\Gamma_q(\beta+|\vec{n}|+1+s)}{\Gamma_q(\beta+|\vec{n}|+1)\Gamma_q(s+1)} - \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)}. \quad (4.4)
\end{aligned}$$

Using recursively relation (4.1) involving the product of r difference operators $\mathcal{M}_{q,n_1}^{\alpha_1}, \dots, \mathcal{M}_{q,n_r}^{\alpha_r}$ acting on the function $f(s) = \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)}$, we have

$$\begin{aligned}
& q^{|\vec{n}|} \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} x(s) \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \\
&= q^{1/2} \sum_{i=1}^r \prod_{j \neq i}^r \frac{\alpha_i q^{|\vec{n}|} - \alpha_j q^{n_j}}{\alpha_i q^{n_i} - \alpha_j q^{n_j}} \frac{x(n_i)(\alpha_i q^{n_i} - 1)(\alpha_j q^{|\vec{n}|+\beta+n_j} - 1)}{\prod_{\nu=1}^r (\alpha_\nu q^{|\vec{n}|+\beta} - 1)} \\
&\times \prod_{l=1}^r \mathcal{M}_{q,n_l-\delta_{l,i}}^{\alpha_l} \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} + \left(\prod_{i=1}^r \frac{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1}{\alpha_i q^{|\vec{n}|+\beta} - 1} x(s) - \sum_{i=1}^r q^{|\vec{n}|i} x(n_i) \right. \\
&\times \frac{\alpha_i q^{\sum_{j=i}^r n_j} - 1}{\alpha_i q^{|\vec{n}|+\beta} - 1} - (q-1)q^{|\vec{n}|+\beta} \prod_{i=1}^r \frac{\alpha_i q^{n_i} - 1}{\alpha_i q^{|\vec{n}|+\beta} - 1} + (q-1) \prod_{i=1}^r x(n_i) \sum_{i=1}^r \prod_{j=1}^i \frac{1}{\alpha_j q^{|\vec{n}|+\beta} - 1} \left. \right) \\
&\times \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)}. \quad (4.5)
\end{aligned}$$

Hence, by using expressions (4.4), (4.5) one gets

$$\begin{aligned}
& x(s) \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \\
&= q^{|\vec{n}|-1/2} \prod_{i=1}^r \frac{\alpha_i q^{|\vec{n}|+\beta} - 1}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \frac{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}{\alpha_k q^{|\vec{n}|+\beta+n_k+1} - 1} \mathcal{M}_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha}} \frac{\Gamma_q(\beta+|\vec{n}|+1+s)}{\Gamma_q(\beta+|\vec{n}|+1)\Gamma_q(s+1)}
\end{aligned}$$

$$\begin{aligned}
& + \prod_{i=1}^r \frac{\alpha_i q^{|\vec{n}|+\beta} - 1}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \left(\sum_{i=1}^r q^{|\vec{n}|+i} x(n_i) \frac{\alpha_i q^{\sum_{j=i}^r n_j} - 1}{\alpha_i q^{|\vec{n}|+\beta} - 1} + (q-1) q^{|\vec{n}|+\beta} \prod_{i=1}^r \frac{\alpha_i q^{n_i} - 1}{\alpha_i q^{|\vec{n}|+\beta} - 1} \right. \\
& \left. - (q-1) \prod_{i=1}^r x(n_i) \sum_{i=1}^r \prod_{j=1}^i \frac{1}{\alpha_j q^{|\vec{n}|+\beta} - 1} + q^{|\vec{n}|} \frac{(\alpha_k q^{n_{k+1}}) x(\beta + |\vec{n}|)}{1 - \alpha_k q^{|\vec{n}|+\beta+n_{k+1}}} \right) \\
& \times \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} \frac{\Gamma_q(\beta + |\vec{n}| + s)}{\Gamma_q(\beta + |\vec{n}|) \Gamma_q(s+1)} - q^{1/2} \sum_{i=1}^r \prod_{j \neq i}^r \frac{\alpha_i q^{|\vec{n}|} - \alpha_j q^{n_j}}{\alpha_i q^{n_i} - \alpha_j q^{n_j}} \frac{x(n_i) (\alpha_i q^{n_i} - 1)}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \\
& \times \prod_{l=1}^r \mathcal{M}_{q,n_l-\delta_{l,i}}^{\alpha_l} \frac{\Gamma_q(\beta + |\vec{n}| + s)}{\Gamma_q(\beta + |\vec{n}|) \Gamma_q(s+1)}.
\end{aligned}$$

As a result of the above calculations one has

$$\begin{aligned}
& x(s) \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} \frac{\Gamma_q(\beta + |\vec{n}| + s)}{\Gamma_q(\beta + |\vec{n}|) \Gamma_q(s+1)} \\
& = q^{|\vec{n}|-1/2} \prod_{i=1}^r \frac{\alpha_i q^{|\vec{n}|+\beta} - 1}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \frac{(\alpha_k q^{n_{k+1}}) x(\beta + |\vec{n}|)}{\alpha_k q^{|\vec{n}|+\beta+n_{k+1}} - 1} \mathcal{M}_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha}} \frac{\Gamma_q(\beta + |\vec{n}| + 1 + s)}{\Gamma_q(\beta + |\vec{n}| + 1) \Gamma_q(s+1)} \\
& + b_{\vec{n},k} \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} \frac{\Gamma_q(\beta + |\vec{n}| + s)}{\Gamma_q(\beta + |\vec{n}|) \Gamma_q(s+1)} - q^{1/2} \sum_{i=1}^r \prod_{j \neq i}^r \frac{\alpha_i q^{|\vec{n}|} - \alpha_j q^{n_j}}{\alpha_i q^{n_i} - \alpha_j q^{n_j}} \frac{x(n_i) (\alpha_i q^{n_i} - 1)}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \\
& \times \frac{1}{\alpha_i q^{|\vec{n}|+\beta+n_i} - 1} \prod_{l=1}^r \mathcal{M}_{q,n_l-\delta_{l,i}}^{\alpha_l} \frac{\Gamma_q(\beta + |\vec{n}| - 1 + s)}{\Gamma_q(\beta + |\vec{n}| - 1) \Gamma_q(s+1)}.
\end{aligned}$$

Finally, multiplying from the left both sides of the previous expression by $\frac{\Gamma_q(\beta) \Gamma_q(s+1)}{\Gamma_q(\beta+s)}$ and using Rodrigues-type formula (2.4) we obtain (4.2), which completes the proof. \square

5 Concluding remarks

In closing, we summarize our findings. We have defined a new family of q -Meixner multiple orthogonal polynomials of the first kind and obtained their explicit expression in terms of Rodrigues-type formula (2.4). We have shown that these multiple orthogonal polynomials are common eigenfunctions of two different $(r+1)$ -order difference operators given in (3.8) and (4.2). By taking limit $q \rightarrow 1$ one recovers the corresponding algebraic relations for multiple Meixner polynomials of the first kind [3]. The expressions (2.4), (3.8), and (4.2) transform into (1.3), (1.4), and (1.5), respectively.

Our algebraic approach for the nearest neighbor recurrence relation (4.2) is purely algebraic and it neither require to introduce type I multiple orthogonality [14] nor an algebraic Riemann-Hilbert approach. Indeed, the q -difference operators involved in the Rodrigues-type formula are the base of the discussed approach.

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